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Singularities in 2D anisotropic potential problems in multi-material corners Real variable approach

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Abstract

An analysis of singular solutions at corners consisting of several different homogeneous wedges is presented for anisotropic potential theory in plane. The concept of transfer matrix is applied for a singularity analysis first of single wedge problems and then of multi-material corner problems. Explicit forms of eigenequations for evaluation of singularity exponent in the case of multi-material corners are derived both for all combinations of homogeneous Neumann and Dirichlet boundary conditions at faces of open corners and for multi-material planes with singular interior points. Perfect transmission conditions at wedge interfaces are considered in both cases. It is proved that singularity exponents are real for open anisotropic multi-material corners, and a sufficient condition for the singularity exponents to be real for anisotropic multi-material planes is deduced. A case of a complex singularity exponent for an anisotropic multi-material plane is reported, apparently for the first time in potential theory. Simple expressions of eigenequations are presented first for open bi-material corners and bi-material planes and second for a crack terminating at a bi-material interface, as examples of application of the theory developed here. Analytical solutions of these eigenequations are presented for interface cracks with any combination of homogeneous boundary conditions along the interface crack faces, and also for a special case of a crack perpendicular to a bi-material interface. A numerical study of variation of the singularity exponent as a function of inclination of a crack terminating at a bi-material interface is presented.

Keywords: Potential theory; Second-order elliptic equation; Anisotropic materials; Corner singularities; Antiplane deformation; Heat conduction; Porous media flow; Electrostatics; FEM; BEM

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1. Introduction

Flux fields as solutions of potential Boundary Value Problems (BVPs), defined by a scalar second-order elliptic equation, may have unbounded values near points where one or more of the following configurations happen: the boundary of the domain is non-smooth, boundary conditions change abruptly or material properties are discontinuous. A generic name of *corners* will be used hereinafter for configurations of this kind of potential BVP. Flux fields whose values increase to infinity will be referred to as *singular at the corner tip*. The knowledge of parameters associated to these singular flux fields can be important for some applications of engineering interest, e.g.: antiplane deformation (Ma and Hour, 1990; Pageau et al., 1995a,b; Ting, 1996), heat transfer, ideal fluid flow and porous media flow (Leguillon and Sanchez-Palencia, 1987; Liggett and Liu, 1983), thermo-elasticity (Ting, 1996; Yosibash, 1998) and electrostatics (Defourny, 1988). It has to be stressed that the singularity analysis of multi-material anisotropic corners is of increasing importance due to their presence in composite materials and electronic devices, where they can be the place of initiation of failure.

The presence of corner singularities in a solution of a BVP can significantly reduce the accuracy of numerical solutions obtained by standard FEM and BEM analysis. To reduce this 'pollution effect', knowledge of the singular solution behaviour may be required in applying either special discretization procedures (e.g. using either elements with singular shape functions or auxiliary mappings, see Georgiu et al., 1996; Helsing, 2000; Lefeber, 1989; Leguillon and Sanchez-Palencia, 1987; Liggett and Liu, 1983; Mera et al., 2002; Oh and Babuška, 1995), or special post-processing procedures (e.g. extraction techniques, see Arad et al., 1998; Yosibash and Szabó, 1995), the latter usually applied after a systematic mesh refinement (e.g. adaptive methods, see Szabó and Babuška, 1991).

It is worth resuming some conclusions of the general theory for analysis of corner singular solutions in elliptic Partial Differential Equations (PDEs) which is at present well-developed (Costabel and Dauge, 1993; Kondratev, 1967). This theory covers the cases of corners with non-homogeneous boundary conditions prescribed along generally curved edges and non-homogeneous materials whose properties are smooth functions of cartesian coordinates. Considering a polar coordinate system (r, θ) centered at the corner tip and applying a separation of variables, the solution basis at the corner is given in terms of power type functions $r^{\lambda} f_{\lambda}(\theta)$, which sometimes have to be completed by power-logarithmic functions $r^{\lambda} \ln r f_{\lambda}(\theta)$. In general, a complex number λ, called *characteristic* or *singularity exponent* (or simply *eigenvalue*), is defined by a solution of a *characteristic equation* (*eigenequation*) associated to the local corner problem configuration. Smooth functions $f_{\lambda}(\theta)$, called *characteristic functions* (*eigenfunctions*), are defined by the solution of the associated eigenvector problem.

The main interest for practical applications represent singularity exponents λ with real part $0 < \mathrm{Re}\,\lambda < 1$. They depend on the *corner tangent sector* and on the constant principal parts, taken at the tip of the corner, of the differential operators in the elliptic PDE and in the boundary conditions. However, they depend neither on the curvature of the corner edges, nor on the smooth right-hand sides of the PDE, nor on the smooth right-hand sides of the boundary conditions prescribed along each face of the corner.

Mathematical analysis of solutions of BVPs in non-smooth domains, where the above discussed singular solutions are present locally at corner tips, can be found in Grisvard (1992) and Nazarov and Plamenevsky (1994).

The present paper deals with the analysis of singular solutions of potential problems at corners formed in general by an arbitrary number of anisotropic homogeneous wedges converging at the corner tip, called *multi-material corners*. The usual case of a corner with two non-bonded edges called corner faces, which are subjected to some boundary conditions, is referred to hereinafter as an *open corner problem*. A particular case of a multi-material corner, when all the plane is occupied by two or more bonded anisotropic wedges converging at an interior point is referred to either as a *closed corner problem* (Dempsey and Sinclair, 1979; Defourny, 1988) or as a *multi-material plane* (Ting, 1997).

Theoretical studies of isotropic multi-material corners by Kellog (1971) and Birkhoff (1972) established the relation between the singularity analysis of these corners and the Sturm-Liouville eigenvalue problem with piecewise constant coefficients. Thus, the usual results on this problem can be extended to the singularity analysis of these corners. Sinclair (1980) considered combinations of four types of homogeneous boundary conditions (Neumann, Dirichlet, convection and radiation) for bi-material isotropic corners and suggested a form of transfer matrix for multi-material corners. He also discussed conditions for existence of a power-logarithmic singularity behaviour. A singularity analysis of isotropic multi-material planes was presented by Liggett and Liu (1983). An elegant analytic approach based on a transfer matrix concept for singularity analysis of isotropic multi-material corners was introduced by Defourny (1988). Explicit analytical expressions of eigenequations and eigenfunctions for isotropic three-material corners were given by Pageau et al. (1995b).

Leguillon and Sanchez-Palencia (1987) studied analytically a particular case of an anisotropic bi-material corner and presented numerical results for anisotropic three-material corners. A complex analogy to the standard Mellin transformation technique was used by Ma and Hour (1989, 1990) for a singularity analysis of open anisotropic bi-material corners and a crack terminating at a straight bi-material interface.

A general numerical procedure, based on discretization of a variational formulation, for analysis of multi-material corner problems was also developed by Leguillon and Sanchez-Palencia. Starting from a Steklov problem formulation Yosibash and Szabó (1995) developed a general numerical approach for singularity analysis in anisotropic multi-material corner problems. A finite element formulation of an eigenvalue problem using special singular finite elements for singularity analysis of anisotropic multi-material corner problems was developed by Pageau et al. (1995a,b). Another procedure, based on a numerical solution of a linear ordinary differential equation with variable coefficients in the case of anisotropic single-material corners was recently presented by Mera et al. (2002).

Singularity analyses of isotropic and anisotropic elastic multi-material corners, respectively, were introduced by Dempsey and Sinclair (1979) (considering all combinations of the basic boundary conditions) and Ting (1997) (considering only combinations of free and fixed edges). Mantič et al. (1997) developed an approach for a singularity analysis of orthotropic elastic single-material corners considering all combinations of basic boundary conditions, which is directly applicable to anisotropic corners as well. Recently Wu (2001) presented a related approach dealing with anisotropic multi-material corners and all combinations of basic boundary conditions except for slipping with friction.

In the present work explicit analytical expressions of eigenequations will be introduced for the case of a multi-material anisotropic open corner consisting of one or more wedges, each one defined by a different material, and with any combination of homogenous Dirichlet and Neumann boundary conditions prescribed at the two corner faces. For simplicity only wedges with straight edges are considered here (see Ting (1996), for a simple approach to deal with curved edges). Additionally, an explicit form of the eigenequation will be presented for the case of a multi-material plane. The eigenequations presented can be solved either analytically, as shown here in some simple cases, or numerically by standard algorithms, like Muller's method. Explicit expressions of $f_\lambda(\theta)$ for each homogeneous wedge will be given as well. The authors of the present work believe that the explicit expressions introduced here will be very useful for analytical study and numerical analysis by FEM and BEM of potential problems where local corner singular solutions appear.

In derivation of the above-mentioned explicit expressions, the concept of *transfer matrix* will be used in a similar way to that developed by Defourny (1988) for an analysis of multi-material isotropic corner problems in potential theory using a real variable approach and by Ting (1997) for an analysis of multi-material anisotropic elastic corners using a complex variable formulation of anisotropic plane elasticity (Clements, 1981; Ting, 1996). In the present work a real variable formulation of anisotropic potential theory, some of whose results were briefly introduced by Mantič and París (1995) and are presented in depth here, will be used. For a complex variable formulation of anisotropic potential theory which could also be used for an analogous singularity analysis as presented here, see Clements (1981) and Ting (1996).

2. Basic equations of anisotropic potential theory

2.1. General solution

Consider a domain $\Omega \subset R^2$ and a two-dimensional cartesian coordinate system x_i $(i = 1, 2)$. The flux vector in an anisotropic potential problem in Ω is defined as

$$h_i(\mathbf{x}) = K_{ij}u_{,j}(\mathbf{x}), \quad i, j = 1, 2, \tag{1}$$

where u is a potential function of $\mathbf{x} = (x_1, x_2) \in \Omega$ and $\mathbf{K}$ is a constant symmetric and positive definite matrix which defines the anisotropic material properties. The vanishing divergence condition for flux vector yields the following second-order elliptic governing equation in terms of the potential function:

$$K_{ij}u_{,ij}(\mathbf{x}) = 0. \tag{2}$$

The normal flux through a line given by a unit normal vector $\mathbf{n}(\mathbf{x})$ is defined as

$$q(\mathbf{x}) = n_i(\mathbf{x})h_i(\mathbf{x}). \tag{3}$$

Let the matrix $\mathbf{L}$ be defined as a factor in a symmetric decomposition of the inverse of the matrix $\mathbf{K}$

$$\mathbf{K}^{-1} = \mathbf{L}^{\mathrm{T}}\mathbf{L}, \quad \mathbf{K} = \mathbf{L}^{-1}(\mathbf{L}^{-1})^{\mathrm{T}}. \tag{4}$$

A general solution of (2) in Ω is given by any harmonic function $\tilde{u}(\tilde{\mathbf{x}})$ in $\widetilde{\Omega} = \mathbf{L}\Omega$ via a transformation of coordinates as follows:

$$u(\mathbf{x}) = \tilde{u}(\tilde{\mathbf{x}}), \quad \text{where } \tilde{\mathbf{x}} = \mathbf{L}\mathbf{x}. \tag{5}$$

Without loss of generality it can be supposed that the determinant $|\mathbf{L}| > 0$. To verify that $(5)_1$ represents a general solution of (2) it is sufficient to substitute $(5)_1$ into (2) and to use $(4)_2$ which leads to

$$K_{ij}u_{,ij}(\mathbf{x}) = \Delta\tilde{u}(\tilde{\mathbf{x}}) = 0. \tag{6}$$

It will be useful to determine some magnitudes associated to the above Laplace equation for $\tilde{u}$ in the transformed coordinates $\tilde{x}_i$ $(i = 1, 2)$.

The transformed flux vector can be evaluated as follows:

$$\tilde{h}_i(\tilde{\mathbf{x}}) = \partial_{\tilde{x}_i}\tilde{u}(\tilde{\mathbf{x}}) = L_{ij}h_j(\mathbf{x}), \tag{7}$$

where (5), the chain rule for differentiation of composite functions, and (1) with (4) have been used in obtaining the final expression.

Consider a line $\Gamma \subset \Omega$ with the unit normal vector at $\mathbf{x} \in \Gamma$ denoted as $\mathbf{n}(\mathbf{x})$. Then, as follows from (7) and from the expression of the unit normal vector $\tilde{\mathbf{n}}(\tilde{\mathbf{x}})$ to the transformed line $\widetilde{\Gamma} = \mathbf{L}\Gamma$ at $\tilde{\mathbf{x}}$ deduced in Appendix A, see (A.2), the transformed normal flux can be expressed as

$$\tilde{q}(\tilde{\mathbf{x}}) = \tilde{n}_i(\tilde{\mathbf{x}})\tilde{h}_i(\tilde{\mathbf{x}}) = \frac{q(\mathbf{x})}{n_K(\mathbf{x})}, \quad \text{where } n_K(\mathbf{x}) = \sqrt{\mathbf{n}^{\mathrm{T}}(\mathbf{x})\mathbf{K}\mathbf{n}(\mathbf{x})}. \tag{8}$$

Note that the matrix $\mathbf{L}$ is not defined uniquely. The list of coordinate transformations used in the literature includes those associated to: the complex variable formulation of anisotropic potential theory by Clements (1981), Choleski triangle (see Golub and Van Loan, 1991) of $\mathbf{K}^{-1}$ by Leguillon and Sanchez-

Palencia (1987) (both in fact closely related to each other), and the principal axes of material orthotropy by Liggett and Liu (1983).

An expression of **L** obtained by the symmetric real Schur decomposition (see Golub and Van Loan, 1991) of $\mathbf{K}^{-1}$ is given in Appendix B, where some basic properties of this transformation are also derived. This decomposition is defined by a rotation of the coordinate system x_i by an angle α to the principal orthotropy axes of **K** (see Liggett and Liu, 1983) by using an orthogonal matrix given by the eigenvectors of **K**, and followed by a scaling of these new coordinate axes by inverse square roots of the eigenvalues, k_1 and k_2 of **K** ($k_1 \geqslant k_2 > 0$). Hence, this transformation seems to be most naturally related to the material structure.

It is worth noting that the only difference between two different **L** transformations is a rotation by a fixed angle given by an orthogonal matrix. For the proof see Appendix C.

2.2. Polar coordinate systems in the real and transformed planes

Consider polar coordinate systems (r, θ) and $(\tilde{r}, \tilde{\theta})$ centered at the origin of cartesian coordinates and defined in the usual way $(x_1, x_2) = (r\cos\theta, r\sin\theta)$ and $(\tilde{x}_1, \tilde{x}_2) = (\tilde{r}\cos\tilde{\theta}, \tilde{r}\sin\tilde{\theta})$. Starting from $(5)_2$ it can be easily shown that $\tilde{\theta}$ is independent of r. Hence

$$\tilde{r} = \tilde{r}(r, \theta), \quad \tilde{\theta} = \tilde{\theta}(\theta). \tag{9}$$

Let us define a unit vector as $\mathbf{e}(\theta) = (\cos\theta, \sin\theta)$. Then the radius vectors can be written as $\mathbf{x} = r\mathbf{e}(\theta)$ and $\tilde{\mathbf{x}} = \tilde{r}\mathbf{e}(\tilde{\theta})$.

It is important to observe that the length of the transformed radius vector $\tilde{\mathbf{x}}$ is independent of a particular choice of **L** because it is given directly by $\mathbf{K}^{-1}$

$$\tilde{r} = \sqrt{\mathbf{x}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{x}} = r\sqrt{\mathbf{e}(\theta)^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{e}(\theta)} = r\rho(\theta), \tag{10}$$

implying the following general expression of the radial scaling factor

$$\rho(\theta) = \sqrt{\frac{K_{11}\sin^2\theta - 2K_{12}\sin\theta\cos\theta + K_{22}\cos^2\theta}{|\mathbf{K}|}} > 0, \tag{11}$$

which is a smooth periodic function of θ with the period π. An equivalent form of $\rho(\theta)$ is given in Appendix B by (B.7).

Although the value of the transformed angle $\tilde{\theta} = \tilde{\theta}(\theta)$ depends on the particular choice of **L**, its derivative is independent of this choice as is shown directly in Appendix D, where a general expression of $\mathrm{d}\tilde{\theta}/\mathrm{d}\theta$ independent of **L** is derived, see (D.3).

Note that the above mentioned fact that $\mathrm{d}\tilde{\theta}/\mathrm{d}\theta$ is independent of a particular choice of **L** is in agreement with the result of Appendix C, which implies that for two different transformations defined by $\mathbf{L}_1$ and $\mathbf{L}_2$, the relation between the associated transformed angles is

$$\tilde{\theta}_2(\theta) = \tilde{\theta}_1(\theta) + \beta, \tag{12}$$

where β is a fixed angle defined by the orthogonal matrix $\mathbf{Q}(\beta)$ in (C.4).

2.3. Wedge and transformed wedge

Consider an anisotropic homogeneous wedge domain $\Omega \subset R^2$ of the interior angle $0 < \omega \leqslant 2\pi$ with the tip at the origin of coordinates and determined by two angles of its straight edges θ_0 and θ_1, $\omega = \theta_1 - \theta_0$,

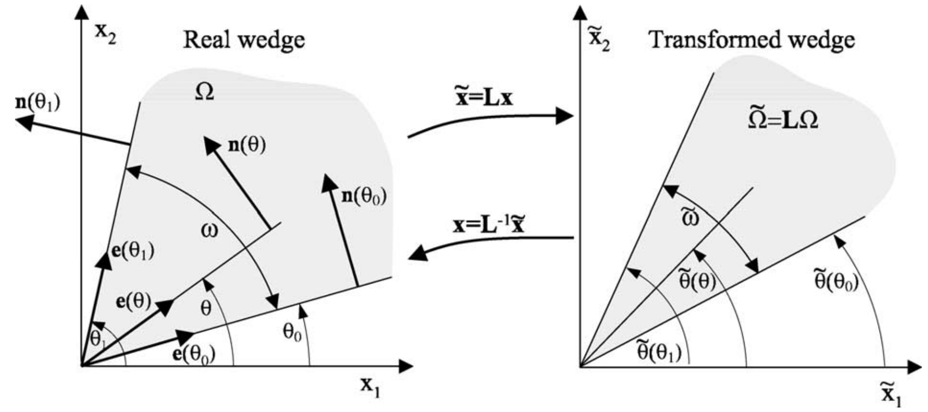

Fig. 1. Single wedge.

see Fig. 1. Thus, $\Omega = \{\mathbf{x} \in R^2 | 0 < r < R(\theta), \theta_0 < \theta < \theta_1\}$, where $R(\theta)$ is either a bounded continuous function or infinity. Using (9), the value of the angle $\tilde{\omega}$ of the transformed wedge $\tilde{\Omega} = \mathbf{L}\Omega$ is given by the integral

$$\tilde{\omega} = \tilde{\theta}(\theta_1) - \tilde{\theta}(\theta_0) = \int_{\theta_0}^{\theta_1} \frac{\mathrm{d}\tilde{\theta}}{\mathrm{d}\theta}\,\mathrm{d}\theta. \tag{13}$$

Substituting (11) into (D.3) and integrating following (13) leads to

$$\tilde{\omega} = \left(\left[\frac{\theta}{\pi} + \frac{1}{2} \right] \pi + \operatorname{arctg} \frac{K_{11}\operatorname{tg}\theta - K_{12}}{\sqrt{|\mathbf{K}|}} \right) \Bigg|_{\theta_0}^{\theta_1}, \tag{14}$$

where $[\cdot]$ means the integer part of a real number. For $\theta = (2k+1)\pi/2$ (k an integer number, $k \in Z$) the undetermined expression of the primitive function in (14) is replaced by its limit value given directly by the value of θ. An analogous expression for $\tilde{\omega}$ can also be directly obtained from (B.8).

It is useful to note that $\tilde{\omega}$ can also be evaluated in a conceptually different way, considering a geometric representation of the wedge, Fig. 1. An application of basic properties of the scalar and vector products to the unit tangential vectors to the wedge edges yields the following elegant expression, Mantič and París (1995):

$$\tilde{\omega} = \pi + \operatorname{sgn}(\omega - \pi) \arccos\left(-\frac{\mathbf{e}^{\mathrm{T}}(\theta_1)\mathbf{K}^{-1}\mathbf{e}(\theta_0)}{\rho(\theta_1)\rho(\theta_0)} \right), \tag{15}$$

where sgn means the usual signum function.

It should be remarked that $\tilde{\omega}$ is not determined only by the wedge angle ω but also by its orientation, i.e. by θ_0 and θ_1. As follows from the above expressions, $\tilde{\omega} = \omega$ for the limit values $\omega = 0$ or 2π and also for $\omega = \pi$. If $\omega = \pi/2$ or $3\pi/2$ and the wedge faces are parallel to the orthotropy axes of the material, then $\tilde{\omega} = \omega$ as well.

3. Single-wedge singularity analysis by a transfer matrix

The solution of (2) in an anisotropic homogeneous wedge domain Ω (Fig. 1) is considered here in a neighbourhood of the wedge tip in the following form of an asymptotic expansion in terms of basis functions, whose coefficients A_l are called generalized flux intensity factors,

$$u(r,\theta)=\sum_{l=0}^{\infty}A_l r^{\lambda_l} f_{\lambda_l}(\theta). \tag{16}$$

In what follows one basis function of the above asymptotic expansion of the form

$$u(r,\theta)=r^{\lambda} f_{\lambda}(\theta), \tag{17}$$

is analysed, where λ is a singularity exponent and $f_{\lambda}(\theta)$ is at the moment an unknown characteristic function defined by the parameter λ. From (5) and results of Section 2.2 it follows that (17) can be written as

$$u(r,\theta)=\tilde{u}(\tilde{r},\tilde{\theta})=\tilde{r}^{\lambda}\tilde{f}_{\lambda}(\tilde{\theta})=r^{\lambda}\rho^{\lambda}(\theta)\tilde{f}_{\lambda}(\tilde{\theta}) \quad \text{thus } f_{\lambda}(\theta)=\rho^{\lambda}(\theta)\tilde{f}_{\lambda}(\tilde{\theta}), \tag{18}$$

where $\tilde{\theta}=\tilde{\theta}(\theta)$.

The normal flux through a straight radial line defined by an angle θ and associated to the normal vector $\mathbf{n}(\theta)=(-\sin\theta,\cos\theta)$, see Fig. 1, is given, in view of (8), by

$$q(r,\theta)=\tilde{q}(\tilde{r},\tilde{\theta})n_K(\theta), \tag{19}$$

where using $(8)_2$ and (11) it can be shown that

$$n_K(\theta)=\sqrt{\mathbf{n}^{\mathrm{T}}(\theta)\mathbf{K}\mathbf{n}(\theta)}=\sqrt{|\mathbf{K}|}\rho(\theta). \tag{20}$$

In the transformed isotropic potential problem the flux through a radial line at $\tilde{\theta}$ is given by

$$\tilde{q}(\tilde{r},\tilde{\theta})=\frac{1}{\tilde{r}}\frac{\partial\tilde{u}(\tilde{r},\tilde{\theta})}{\partial\tilde{\theta}}. \tag{21}$$

Hence, applying (18)–(21) for $\lambda\neq 0$,

$$q(r,\theta)=\sqrt{|\mathbf{K}|}\frac{1}{r}\frac{\partial\tilde{u}(\tilde{r},\tilde{\theta})}{\partial\tilde{\theta}}=r^{\lambda-1}\rho^{\lambda}(\theta)\sqrt{|\mathbf{K}|}\frac{\mathrm{d}\tilde{f}_{\lambda}(\tilde{\theta})}{\mathrm{d}\tilde{\theta}}, \tag{22}$$

which implies that $q(r,\theta)$ is singular for $r\to 0^+$ if $\mathrm{Re}\,\lambda<1$. Note that potential solutions with a finite energy in a bounded domain Ω correspond to $\mathrm{Re}\,\lambda>0$.

Define a new function $\phi(r,\theta)$, called flux function, which fulfills

$$q(r,\theta)=\frac{\partial\phi(r,\theta)}{\partial r}, \tag{23}$$

in the following way, for $\lambda\neq 0$,

$$\phi(r,\theta)=\frac{r^{\lambda}}{\lambda}\rho^{\lambda}(\theta)\sqrt{|\mathbf{K}|}\frac{\mathrm{d}\tilde{f}_{\lambda}(\tilde{\theta})}{\mathrm{d}\tilde{\theta}}. \tag{24}$$

Note that vanishing of $q(r,\theta)$ defined by (22) is equivalent to vanishing of $\phi(r,\theta)$ defined by (24).

From the form of Laplace equation in polar coordinates

$$\Delta\tilde{u}(\tilde{r},\tilde{\theta})=\left(\partial_{\tilde{r}}^2+\tilde{r}^{-1}\partial_{\tilde{r}}+\tilde{r}^{-2}\partial_{\tilde{\theta}}^2\right)\tilde{u}=0 \tag{25}$$

it is directly obtained by substitution of $(18)_1$ that

$$\Delta\tilde{u}(\tilde{r},\tilde{\theta})=\tilde{r}^{\lambda-2}\left(\frac{\mathrm{d}^2\tilde{f}_{\lambda}(\tilde{\theta})}{\mathrm{d}\tilde{\theta}^2}+\lambda^2\tilde{f}_{\lambda}(\tilde{\theta})\right)=0 \tag{26}$$

and consequently a general solution for $\tilde{f}_{\lambda}(\tilde{\theta})$ takes the following form including two constants a_c and a_s:

$$\tilde{f}_\lambda(\tilde{\theta}) = a_c \cos \lambda\tilde{\theta} + a_s \sin \lambda\tilde{\theta}. \tag{27}$$

Then, the pair of potential and flux function solutions, for $\lambda \neq 0$, can be written as a vector

$$\mathbf{w}(r,\theta) = \begin{bmatrix} u(r,\theta) \\ \phi(r,\theta) \end{bmatrix} = r^\lambda \rho^\lambda(\theta) \begin{bmatrix} \cos \lambda\tilde{\theta} & \sin \lambda\tilde{\theta} \\ -\sqrt{|\mathbf{K}|}\sin \lambda\tilde{\theta} & \sqrt{|\mathbf{K}|}\cos \lambda\tilde{\theta} \end{bmatrix} \begin{bmatrix} a_c \\ a_s \end{bmatrix} \tag{28}$$

or in a compact form

$$\mathbf{w}(r,\theta) = r^\lambda \mathbf{Z}(\lambda,\theta)\mathbf{a}, \tag{29}$$

where $\mathbf{a}^{\mathrm{T}} = [a_c, a_s]$ and $\tilde{\theta} = \tilde{\theta}(\theta)$. Note that $|\mathbf{Z}(\lambda,\theta)| = \rho^{2\lambda}(\theta)\sqrt{|\mathbf{K}|} \neq 0$. Thus $\mathbf{Z}(\lambda,\theta)$ is a regular matrix.

Equation (29) for $\theta = \theta_0$ and θ_1 gives

$$\mathbf{w}(r,\theta_0) = r^\lambda \mathbf{Z}(\lambda,\theta_0)\mathbf{a} \quad \text{and} \quad \mathbf{w}(r,\theta_1) = r^\lambda \mathbf{Z}(\lambda,\theta_1)\mathbf{a}. \tag{30}$$

Insertion of $\mathbf{a}$ obtained from $(30)_1$ into $(30)_2$ leads for $\lambda \neq 0$ to

$$\mathbf{w}(r,\theta_1) = \mathbf{E}(\lambda,\theta_1,\theta_0)\mathbf{w}(r,\theta_0), \tag{31}$$

where

$$\begin{aligned} \mathbf{E}(\lambda,\theta_1,\theta_0) = \mathbf{Z}(\lambda,\theta_1)\mathbf{Z}^{-1}(\lambda,\theta_0) &= \left(\frac{\rho(\theta_1)}{\rho(\theta_0)}\right)^\lambda \begin{bmatrix} \cos \lambda\tilde{\omega} & \frac{1}{\sqrt{|\mathbf{K}|}}\sin \lambda\tilde{\omega} \\ -\sqrt{|\mathbf{K}|}\sin \lambda\tilde{\omega} & \cos \lambda\tilde{\omega} \end{bmatrix} \\ &= \begin{bmatrix} E^{(1)}(\lambda) & E^{(2)}(\lambda) \\ E^{(3)}(\lambda) & E^{(4)}(\lambda) \end{bmatrix}, \end{aligned} \tag{32}$$

$\tilde{\omega} = \tilde{\theta}(\theta_1) - \tilde{\theta}(\theta_0)$ having been evaluated in Section 2.3.

The 2×2 matrix $\mathbf{E}$ is the transfer matrix defined in an analogous way to Defourny (1988) and Ting (1997). It transfers the value of $\mathbf{w}$ at θ_0 to $\mathbf{w}$ at θ_1. Elements of this matrix are denoted here as in Ting (1997).

It can be seen that

$$E^{(1)}(\lambda) = E^{(4)}(\lambda) \quad \text{and} \quad E^{(3)}(\lambda) = -|\mathbf{K}|E^{(2)}(\lambda). \tag{33}$$

It is also useful to observe that

$$|\mathbf{E}| = (\rho(\theta_1)/\rho(\theta_0))^{2\lambda} \neq 0, \tag{34}$$

and for a particular case of an isotropic material of wedge $|\mathbf{E}| = 1$, cf. Defourny (1988).

Eq. (31) can be partitioned in the following way:

$$u(r,\theta_1) = E^{(1)}(\lambda)u(r,\theta_0) + E^{(2)}(\lambda)\phi(r,\theta_0), \tag{35}$$

$$\phi(r,\theta_1) = E^{(3)}(\lambda)u(r,\theta_0) + E^{(4)}(\lambda)\phi(r,\theta_0). \tag{36}$$

Recall that (35) and (36) hold only for $u(r,\theta)$ and $\phi(r,\theta)$, respectively, in the forms (17) and (24).

Consider now four configurations of homogeneous Neumann (N) and Dirichlet (D) boundary conditions at wedge edges and potential solutions of the form (17) for $\lambda \neq 0$. System (35) and (36) have, for each configuration, a non-trivial solution if and only if λ is a root of the corresponding eigenequation. Eigenequations (in a general form and also in a simple explicit form) and eigenvalues for these homogeneous wedge problems are summarized in Table 1, where Z is the set of integer numbers.

Note that in the N–N case the solution for $\lambda_0 = 0$ corresponds to a constant solution, $u(r,\theta) = a_c$ and $q(r,\theta) = \phi(r,\theta) = 0$.

As can be observed, the roots of the eigenequations presented are real and simple, and coincide in the N–N and D–D cases because of $(33)_2$ and in the N–D and D–N cases because of $(33)_1$.

Table 1
Eigenequations, eigenvalues and general singular solutions for homogeneous single wedge problems

Case	Boundary conditions	Eigenequation		Eigen-values λ_l ($l \in Z$)	Series expansion of non-vanishing either potential or flux at θ_0	Potential solution $u(r,\theta)$ ($\theta_0 \leqslant \theta \leqslant \theta_1$)
N–N	$q(r,\theta_0) = q(r,\theta_1) = 0$	$E^{(3)}(\lambda) = 0$	$\sin\lambda\tilde{\omega} = 0$	$l\frac{\pi}{\tilde{\omega}}$	$u(r,\theta_0) = \sum_{l=0}^{\infty} A_l r^{\lambda_l} \rho^{\lambda_l}(\theta_0)$	$\sum_{l=0}^{\infty} A_l r^{\lambda_l} \rho^{\lambda_l}(\theta) \times \cos\left(\lambda_l(\tilde{\theta}(\theta) - \tilde{\theta}(\theta_0))\right)$
D–D	$u(r,\theta_0) = u(r,\theta_1) = 0$	$E^{(2)}(\lambda) = 0$	$\sin\lambda\tilde{\omega} = 0$	$l\frac{\pi}{\tilde{\omega}}$	$q(r,\theta_0) = \sqrt{\lvert\mathbf{K}\rvert}\sum_{l=1}^{\infty} A_l r^{\lambda_l - 1} \rho^{\lambda_l}(\theta_0)\lambda_l$	$\sum_{l=1}^{\infty} A_l r^{\lambda_l} \rho^{\lambda_l}(\theta) \times \sin\left(\lambda_l(\tilde{\theta}(\theta) - \tilde{\theta}(\theta_0))\right)$
N–D	$q(r,\theta_0) = u(r,\theta_1) = 0$	$E^{(1)}(\lambda) = 0$	$\cos\lambda\tilde{\omega} = 0$	$\left(l + \frac{1}{2}\right)\frac{\pi}{\tilde{\omega}}$	$u(r,\theta_0) = \sum_{l=0}^{\infty} A_l r^{\lambda_l} \rho^{\lambda_l}(\theta_0)$	$\sum_{l=0}^{\infty} A_l r^{\lambda_l} \rho^{\lambda_l}(\theta) \times \cos\left(\lambda_l(\tilde{\theta}(\theta) - \tilde{\theta}(\theta_0))\right)$
D–N	$u(r,\theta_0) = q(r,\theta_1) = 0$	$E^{(4)}(\lambda) = 0$	$\cos\lambda\tilde{\omega} = 0$	$\left(l + \frac{1}{2}\right)\frac{\pi}{\tilde{\omega}}$	$q(r,\theta_0) = \sqrt{\lvert\mathbf{K}\rvert}\sum_{l=0}^{\infty} A_l r^{\lambda_l - 1} \rho^{\lambda_l}(\theta_0)\lambda_l$	$\sum_{l=0}^{\infty} A_l r^{\lambda_l} \rho^{\lambda_l}(\theta) \times \sin\left(\lambda_l(\tilde{\theta}(\theta) - \tilde{\theta}(\theta_0))\right)$

Explicit forms of general singular potential solutions $u(r,\theta)$ at a single wedge with any configuration of homogeneous boundary conditions can be obtained using expression (31) written for any angle θ, $\theta_0 \leqslant \theta \leqslant \theta_1$, instead of only its maximum value θ_1. Hence,

$$\mathbf{w}(r,\theta) = \mathbf{E}(\lambda,\theta,\theta_0)\mathbf{w}(r,\theta_0). \tag{37}$$

Starting from suitable series expansions of a non-vanishing distribution of either potential or flux at θ_0, expressions of general singular solutions $u(r,\theta)$ are presented in Table 1 as well. Analogous expressions for $q(r,\theta)$ can be obtained in a similar way using (37).

4. Multi-material corner singularity analysis

Consider a corner domain $\Omega \subset R^2$ that consists of $m \geqslant 2$ different anisotropic homogeneous wedge domains $\Omega_n = \{\mathbf{x} \in R^2 | 0 < r < R(\theta), \theta_{n-1} < \theta < \theta_n\}$ of angle $\omega_n = \theta_n - \theta_{n-1}$ $(n = 1,\ldots,m)$. Material properties of the nth wedge are defined by a symmetric positive definite matrix $\mathbf{K}_n$. All wedges refer to the same cartesian and polar coordinate systems, see Fig. 2. The total corner angle ψ is

$$0 < \psi = \theta_m - \theta_0 = \omega_1 + \omega_2 + \cdots + \omega_m \leqslant 2\pi. \tag{38}$$

Considering the same singularity exponent λ of a basis function $u(r,\theta) = r^\lambda f_\lambda(\theta)$ for all wedges, the key idea, following Defourny (1988) and Ting (1997), is to apply equations derived in Section 3, basically Eq. (31), to each wedge of the multi-material corner together with an application of the perfect transmission conditions at wedge interfaces.

Denoting by a subscript n the quantities associated to the nth wedge, Eq. (31) is written as

$$\mathbf{w}_n(r,\theta_n) = \mathbf{E}_n(\lambda,\theta_n,\theta_{n-1})\mathbf{w}_n(r,\theta_{n-1}), \tag{39}$$

where from (32)

$$\mathbf{E}_n(\lambda,\theta_n,\theta_{n-1}) = \left(\frac{\rho_n(\theta_n)}{\rho_n(\theta_{n-1})}\right)^\lambda \begin{bmatrix} \cos\lambda\tilde{\omega}_n & \frac{1}{\sqrt{\lvert\mathbf{K}_n\rvert}}\sin\lambda\tilde{\omega}_n \\ -\sqrt{\lvert\mathbf{K}_n\rvert}\sin\lambda\tilde{\omega}_n & \cos\lambda\tilde{\omega}_n \end{bmatrix}. \tag{40}$$

Transmission conditions across the wedge interface at θ_n demand that

$$\mathbf{w}_n(r,\theta_n^-) = \mathbf{w}_{n+1}(r,\theta_n^+), \quad r > 0, \tag{41}$$

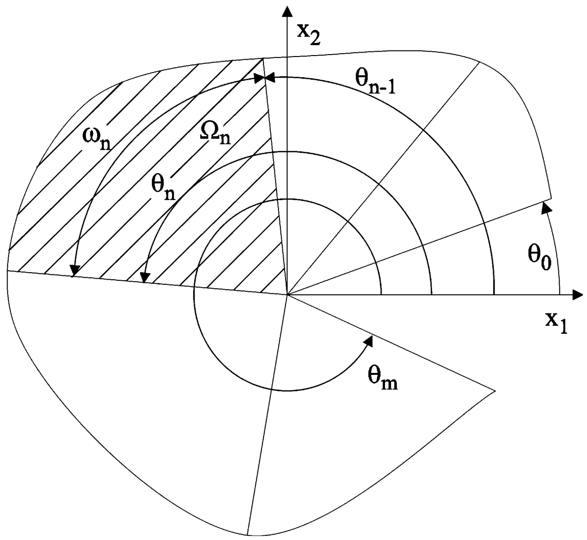

Fig. 2. Multi-material corner.

where θ_n^- and θ_n^+, respectively, denote unilateral limits from the left and right. Eq. (41) means continuity of potential and normal flux (or flux function) associated to the normal vector $\mathbf{n}(\theta_n) = (-\sin\theta_n, \cos\theta_n)$.

Repeated application of (39) and (41) for $n = 1, \ldots, m$ leads to

$$\mathbf{w}_m(r, \theta_m) = \mathbf{C}_m(\lambda, \theta_m, \theta_0)\mathbf{w}_1(r, \theta_0), \tag{42}$$

where

$$\mathbf{C}_m(\lambda, \theta_m, \theta_0) = \mathbf{E}_m(\lambda, \theta_m, \theta_{m-1})\mathbf{E}_{m-1}(\lambda, \theta_{m-1}, \theta_{m-2}) \cdots \mathbf{E}_1(\lambda, \theta_1, \theta_0) = \begin{bmatrix} C_m^{(1)}(\lambda) & C_m^{(2)}(\lambda) \\ C_m^{(3)}(\lambda) & C_m^{(4)}(\lambda) \end{bmatrix}. \tag{43}$$

Using (34) and (43) it is obtained that

$$|\mathbf{C}_m(\lambda, \theta_m, \theta_0)| = \prod_{n=1}^{m} \left(\frac{\rho_n(\theta_n)}{\rho_n(\theta_{n-1})} \right)^{2\lambda} \neq 0, \tag{44}$$

which implies a useful relation

$$C_m^{(1)}(\lambda)C_m^{(4)}(\lambda) \neq C_m^{(2)}(\lambda)C_m^{(3)}(\lambda). \tag{45}$$

Eq. (44) reduces to $|\mathbf{C}_m(\lambda, \theta_m, \theta_0)| = 1$, for isotropic materials, cf. Defourny (1988).

For the case of a multi-material plane, when $\psi = 2\pi$, and edges at $\theta = \theta_m$ and $\theta = \theta_0$ correspond to the same line, the transmission condition across this line demands

$$\mathbf{w}_m(r, \theta_m^-) = \mathbf{w}_1(r, \theta_0^+), \tag{46}$$

which implies, using (42), that

$$(\mathbf{C}_m(\lambda, \theta_m, \theta_0) - \mathbf{I})\mathbf{w}_1(r, \theta_0) = \mathbf{0}, \tag{47}$$

where $\mathbf{I}$ is the 2×2 identity matrix. A non-trivial solution for $\mathbf{w}_1(r, \theta_0)$ exists if

$$|\mathbf{C}_m(\lambda, \theta_m, \theta_0) - \mathbf{I}| = 0, \tag{48}$$

Table 2
Eigenequations for open multi-material corners

Case	Boundary conditions	Eigenequation
N–N	$q_1(r,\theta_0) = q_m(r,\theta_m) = 0$	$C_m^{(3)}(\lambda) = 0$
D–D	$u_1(r,\theta_0) = u_m(r,\theta_m) = 0$	$C_m^{(2)}(\lambda) = 0$
N–D	$q_1(r,\theta_0) = u_m(r,\theta_m) = 0$	$C_m^{(1)}(\lambda) = 0$
D–N	$u_1(r,\theta_0) = q_m(r,\theta_m) = 0$	$C_m^{(4)}(\lambda) = 0$

which is equivalent to

$$C_m^{(1)}(\lambda) + C_m^{(4)}(\lambda) = 1 + |\mathbf{C}_m(\lambda, \theta_m, \theta_0)|. \tag{49}$$

The roots λ of this equation represent singularity exponents of the closed corner problem.

For the case of a multi-material corner in which homogeneous Neumann or Dirichlet boundary conditions are prescribed on lines at $\theta = \theta_0$ and $\theta = \theta_m$, the analysis is similar to the one presented in Section 3. Using analogous equations to (35) and (36)

$$u_m(r, \theta_m) = C_m^{(1)}(\lambda)u_1(r, \theta_0) + C_m^{(2)}(\lambda)\phi_1(r, \theta_0), \tag{50}$$

$$\phi_m(r, \theta_m) = C_m^{(3)}(\lambda)u_1(r, \theta_0) + C_m^{(4)}(\lambda)\phi_1(r, \theta_0), \tag{51}$$

the corresponding eigenequations for different combinations of homogeneous boundary conditions can be derived, see Table 2.

Due to the fact that an analogous relation to $(33)_2$ does not hold for matrix $\mathbf{C}_m(\lambda, \theta_m, \theta_0)$, the roots of eigenequations in the N–N and D–D cases are in general different, the same holding for roots of eigenequations in the N–D and D–N cases. Additionally, from (45) it follows that if λ is an eigenvalue for the N–N or D–D cases it cannot be any eigenvalue either for N–D or D–N, and viceversa.

An elementary proof that singularity exponents of open and closed isotropic multi-material corners (roots of eigenequations given in Table 2 and (49)) are real numbers is presented in Appendix E (see also a discussion on this topic by Kellog (1971) and Birkhoff (1972)). In view of the real character of singularity exponents in isotropic corners, it is shown in Appendix F that singularity exponents of any open anisotropic multi-material corner are also real. In a particular case of open anisotropic bi-material corners, this fact was already proved by Leguillon and Sanchez-Palencia (1987) and Ma and Hour (1989). Ma and Hour (1990) also showed that singularity exponents are real for a crack terminating at an interface. Nevertheless, the same fact is proved for closed anisotropic multi-material corners in Appendix F only if condition (F.4) holds. An example of a closed bi-material corner, which does not fulfill this condition, with complex singularity exponents will be presented in the next section.

An expression of the solution in the nth wedge, corresponding to a particular value of λ as a root of an eigenequation given in Table 2 or (49) can be obtained as in (37) using the representation

$$\mathbf{w}_n(r, \theta) = \mathbf{C}_n(\lambda, \theta, \theta_0)\mathbf{w}_1(r, \theta_0), \quad \theta_{n-1} \leqslant \theta \leqslant \theta_n, \tag{52}$$

where $\mathbf{w}_1(r, \theta_0)$ is determined from the boundary condition at θ_0 for open corner problems or as an eigenvector in (47) for closed corner problems.

5. Examples

Two typical examples of multi-material corners appearing very frequently in engineering practice are studied in what follows. They allow us, first to corroborate the present general approach comparing with the corresponding results obtained by other authors using more specific approaches, and second to present some new results which are important from an engineering point of view.

The simplest case of multi-material corners, bi-material corners, is studied first. These corners can model many local configurations present in composite materials, electronic packaging, etc.

The second example, a crack terminating at a bi-material interface, has been motivated by its typical presence in composite materials and coatings (e.g. thermal barrier coatings). The configuration analysed can be considered as a simple model of a failure mode in such applications. For an extensive bibliography on this configuration see Ma and Hour (1990).

5.1. Bi-material corners

Consider the case of two adjacent anisotropic wedges ($m = 2$). Then applying (40) and (43)

$$\mathbf{C}_2(\lambda, \theta_2, \theta_0) = \left(\frac{\rho_2(\theta_2)}{\rho_2(\theta_1)} \frac{\rho_1(\theta_1)}{\rho_1(\theta_0)} \right)^{\lambda} \times \begin{bmatrix} \cos\lambda\tilde{\omega}_2 \cos\lambda\tilde{\omega}_1 - \sqrt{\frac{|\mathbf{K}_1|}{|\mathbf{K}_2|}} \sin\lambda\tilde{\omega}_2 \sin\lambda\tilde{\omega}_1 & \frac{1}{\sqrt{|\mathbf{K}_1|}} \cos\lambda\tilde{\omega}_2 \sin\lambda\tilde{\omega}_1 + \frac{1}{\sqrt{|\mathbf{K}_2|}} \sin\lambda\tilde{\omega}_2 \cos\lambda\tilde{\omega}_1 \\ -\sqrt{|\mathbf{K}_2|} \sin\lambda\tilde{\omega}_2 \cos\lambda\tilde{\omega}_1 - \sqrt{|\mathbf{K}_1|} \cos\lambda\tilde{\omega}_2 \sin\lambda\tilde{\omega}_1 & \cos\lambda\tilde{\omega}_2 \cos\lambda\tilde{\omega}_1 - \sqrt{\frac{|\mathbf{K}_2|}{|\mathbf{K}_1|}} \sin\lambda\tilde{\omega}_2 \sin\lambda\tilde{\omega}_1 \end{bmatrix}. \tag{53}$$

Defining a bi-material parameter ϵ, as in Ma and Hour (1989), as

$$\epsilon = \frac{1-\kappa}{1+\kappa}, \quad \kappa = \sqrt{\frac{|\mathbf{K}_1|}{|\mathbf{K}_2|}}, \quad -1 < \epsilon < 1, \tag{54}$$

simple expressions of eigenequations for the case of open bi-material corners can be obtained from (53) and eigenequations given in Table 2, see Table 3.

Eigenequations presented in Table 3 were previously obtained using a complex variable approach in conjunction with the complex Mellin transform by Ma and Hour (1989). Eigenequation for the N–N case was also deduced in a different way by Leguillon and Sanchez-Palencia (1987).

If interface cracks are considered, i.e. $\omega_1 = \omega_2 = \pi$, then $\tilde{\omega}_1 = \tilde{\omega}_2 = \pi$. Explicit values of singularity exponents λ_l for this particular case of open bi-material corners are shown in Table 3 as well. Note that these expressions of λ_l for both N–D and D–N cases can also be written in terms of κ in view of the following relation:

$$\arctan\sqrt{\kappa} = \tfrac{1}{2}\arccos\epsilon. \tag{55}$$

Expression of λ_l for both N–N and D–D cases shown in Table 3 corresponds to a well known result, see e.g. Ma and Hour (1989) and Ting (1986, 1996). An equivalent expression of λ_l for the N–D case to that given in Table 3 was presented for isotropic materials by Ting (1986) and for anisotropic materials by Ma and Hour (1989).

Table 3
Eigenequations for open bi-material corners and eigenvalues for interface cracks as a particular case

Case	Eigenequation for an open bi-material corner	Eigenvalues for an interface crack λ_l ($l \in Z$)
N–N	$\sin\lambda(\tilde{\omega}_1 + \tilde{\omega}_2) = \epsilon \sin\lambda(\tilde{\omega}_1 - \tilde{\omega}_2)$	$\frac{l}{2}$
D–D	$\sin\lambda(\tilde{\omega}_1 + \tilde{\omega}_2) = -\epsilon \sin\lambda(\tilde{\omega}_1 - \tilde{\omega}_2)$	$\frac{l}{2}$
N–D	$\cos\lambda(\tilde{\omega}_1 + \tilde{\omega}_2) = -\epsilon \cos\lambda(\tilde{\omega}_1 - \tilde{\omega}_2)$	$l + \frac{1}{2} \pm \frac{1}{2\pi}\arccos\epsilon$
D–N	$\cos\lambda(\tilde{\omega}_1 + \tilde{\omega}_2) = \epsilon \cos\lambda(\tilde{\omega}_1 - \tilde{\omega}_2)$	$l \pm \frac{1}{2\pi}\arccos\epsilon$

In the case of a bi-material plane, the eigenequation (49) takes the following form obtained from (53)

$$\cos\lambda\left(\tilde{\omega}_1+\tilde{\omega}_2\right)=\epsilon^2\cos\lambda\left(\tilde{\omega}_1-\tilde{\omega}_2\right)+\frac{1-\epsilon^2}{2}\left(\left(\frac{\rho_2(\theta_2)}{\rho_2(\theta_1)}\frac{\rho_1(\theta_1)}{\rho_1(\theta_0)}\right)^{\lambda}+\left(\frac{\rho_2(\theta_2)}{\rho_2(\theta_1)}\frac{\rho_1(\theta_1)}{\rho_1(\theta_0)}\right)^{-\lambda}\right). \quad (56)$$

An equivalent eigenequation to (56) was derived for an isotropic bi-material plane in Liggett and Liu (1983), see Eq. (4.18) there. [1]

As has been mentioned in Section 4, the roots of (56) are not necessarily real if the expression which appears on the left-hand side of (F.4) is different from unity. Observe that this expression appears two times on the right-hand side of (56).

It has to be stressed that a complex singularity exponent ($\lambda=\mathrm{Re}\lambda+\mathrm{iIm}\lambda$) has an associated oscillatory singular behaviour of flux of type $r^{\mathrm{Re}\,\lambda-1}\sin(\mathrm{Im}\,\lambda\ln r)$ or $r^{\mathrm{Re}\,\lambda-1}\cos(\mathrm{Im}\,\lambda\ln r)$. Such solutions are well known in the case of 'open' elastic interface cracks, where no contact zone is considered at the crack tip, e.g. Ting (1986, 1996).

Following an analysis given in Leguillon and Sanchez-Palencia (1987), if a closed bi-material corner has a symmetry axis (taking into account the orientation of orthotropy axes in both materials) then the solutions at this corner can be split into a symmetric and a skew symmetric part, which are, respectively, given as solutions of the Neumann and Dirichlet problems on one half of the bi-material plane with the boundary defined by the symmetry axis of the closed corner. Therefore, the singularity exponents of such a closed corner are defined by the singularity exponents of the Neumann and Dirichlet problems at the corresponding open corner defined by the pertinent half-plane. Consequently these singularity exponents are real, as has been explained previously. It is an easy exercise to check that in this particular case of closed anisotropic corners the condition (F.4) holds.

As follows from the above analysis, the only possible candidates of closed bi-material corners to have complex eigenvalues are those which do not have any symmetry axis. A simple example of such a closed bi-material corner is presented in what follows. Consider $\mathbf{K}_1=\mathbf{I}$, $\mathbf{K}_2=\mathrm{diag}[10,0.1]$, $\theta_0=0°$, $\theta_1=90°$ and $\theta_2=360°$, thus $\omega_1=90°$ and $\omega_2=270°$. The value of the expression on the left-hand side of (F.4) is 0.1. Then, the singularity exponents, roots of (56), with $0<\mathrm{Re}\,\lambda<1$ are $\lambda=0.8816020381\pm0.3230787589i$.

5.2. Crack terminating at a bi-material interface

Consider the case of two perfectly bonded half-planes with material properties defined by $\mathbf{K}_1$ for $x_2>0$ and by $\mathbf{K}_2$ for $x_2<0$. Consider a crack in the top half-plane terminating at the interface, Fig. 3. For the sake of brevity only the case of homogeneous Neumann boundary conditions at the crack lips is analysed here. Note that any combination of boundary conditions can be analysed using the pertinent eigenequation as given in Table 2. Let the angle defined by the crack and the positive axis x_1 be denoted by ω, $0<\omega<\pi$. For the limit values of $\omega=0$ or π the crack changes to an interface crack.

The above configuration can be represented, using the present notation, as a tri-material wedge defined as follows: $\theta_0=\omega$, $\theta_1=\pi$, $\theta_2=2\pi$, $\theta_3=2\pi+\omega$; $\omega_1=\pi-\omega$, $\omega_2=\pi$, $\omega_3=\omega$; $\mathbf{K}_3=\mathbf{K}_1$ and $q_1(r,\theta_0)=q_3(r,\theta_3)=0$.

Then, using (40), (43), the eigenequation for the N–N case in Table 2 takes the following form after some rearrangements:

$$\sin\lambda\pi\left(\kappa\sin\lambda\tilde{\omega}\sin\lambda(\pi-\tilde{\omega})-\kappa^{-1}\cos\lambda\tilde{\omega}\cos\lambda(\pi-\tilde{\omega})-\cos\lambda\pi\right)=0, \quad (57)$$

[1] There is a misprint in the second member of Eq. (4.18) in Liggett and Liu (1983), the correct form of this term being $-2K_1K_2\cos\lambda\delta_1\cos\lambda\delta_2$.

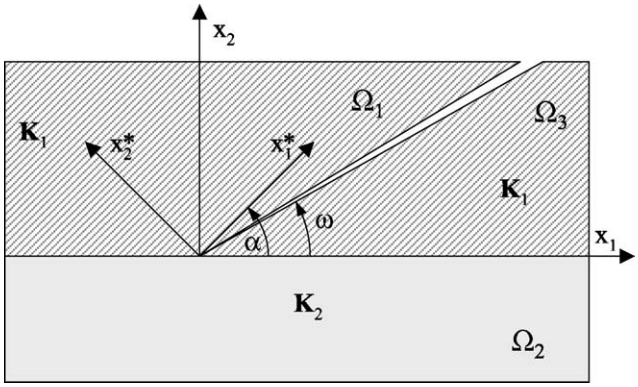

Fig. 3. Crack terminating at a bi-material interface.

κ being defined in (54), and, in view of (15),

$$\tilde{\omega} = \pi - \arccos\left(-\frac{\mathbf{e}^{\mathrm{T}}(\omega)\mathbf{K}_1^{-1}\mathbf{e}(0)}{\rho_1(\omega)\rho_1(0)} \right), \tag{58}$$

with subscript 1 referring to the material in the top half-plane.

Eq. (57) is verified either when the first term vanishes, i.e. $\sin\lambda\pi = 0$, thus $\lambda = l$, $l \in Z$, or when the second term vanishes, which is equivalent to the following equation:

$$\cos\lambda\pi = -\epsilon\cos\lambda(\pi - 2\tilde{\omega}), \tag{59}$$

ϵ being defined in (54). An equivalent eigenequation to (59) was obtained by Ma and Hour (1990) using the complex Mellin integral transform. From a simple analysis of (59), it can be seen that the singularity exponent λ is smaller for negative than for positive values of ϵ, which means that the corresponding flux singular states are more severe when the crack is placed in the material with a higher value of determinant of $\mathbf{K}$, i.e. $|\mathbf{K}_1| > |\mathbf{K}_2|$, c.f. Ma and Hour (1990).

Note that if $|\mathbf{K}_1| = |\mathbf{K}_2|$ then $\epsilon = 0$, and consequently the singularity exponent takes the value $\lambda = 0.5$ for any angle ω and any orientation of the orthotropy axes of both materials.

For a particular case where the orthotropy axes of the material at the top half-plane are parallel and perpendicular to the interface and the crack is perpendicular to the interface it holds that $\tilde{\omega} = \omega = \pi/2$ and an explicit expression of the singularity exponent takes the form

$$\lambda = 1 - \frac{1}{\pi}\arccos\epsilon. \tag{60}$$

Thus, in this case, $\lambda = 0.5$ if and only if $\epsilon = 0$.

Finally, consider the following configurations of material properties (see Fig. 3): $\mathbf{K}_1 = \mathbf{V}\cdot\mathrm{diag}[1,k]\cdot\mathbf{V}^{\mathrm{T}}$, where the material parameter $k = 0.25$ or 4, $\mathbf{V}$, given by expression (B.3) in Appendix B, defines the angle α ($\alpha = 0°, 30°, 90°$) of orientation of material orthotropy axes, and $\mathbf{K}_2 = \mathbf{I}$. Then, the singularity exponents obtained by a numerical solution of (59) are presented in Fig. 4.

As can be observed, the behaviour of the singularity exponent as a function of the crack inclination verifies the expected symmetries, and $\lambda = 0.5$ only for the limit cases of interface cracks. It is also interesting to observe that in the case where $|\mathbf{K}_1| > |\mathbf{K}_2|$ the lowest values of λ are achieved if the direction of the crack approximates to the orthotropy direction associated to the largest eigenvalue of $\mathbf{K}_1$.

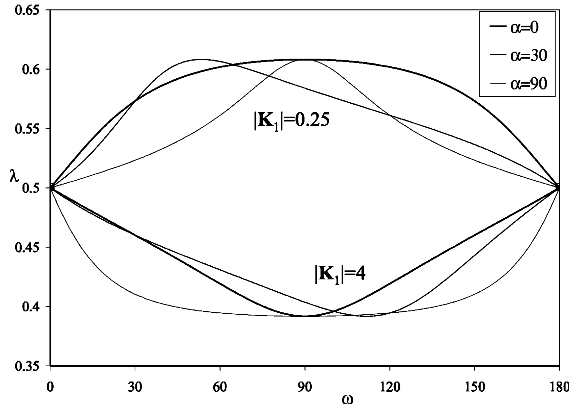

Fig. 4. Singularity exponent as a function of the crack inclination to the interface.

6. Concluding remarks

A new general and powerful method for calculation of eigenequations whose roots define the singularity exponents at multi-material anisotropic corners in potential theory has been developed. Eigenquation computation is reduced to the evaluation of a product of 2×2 matrices, one for each homogeneous wedge. When the corner is open the relevant expression is given by an element of the resulting matrix. When the corner is closed the relevant expression is given by the determinant of the resulting matrix minus the identity matrix.

Proof that singularity exponents are real in open anisotropic multi-material corners has been presented for the first time, to the best authors knowledge. Additionally, a sufficient condition for the singularity exponents to be real in closed anisotropic multi-material corners has been derived. As a consequence of these new theoretical results, the first example in the literature, of a closed anisotropic bi-material corner which violates the above condition and which has associated a complex singularity exponent has been found. Notice that singular solutions corresponding to complex singularity exponents have an oscillatory character, similar to that well-known in some elastic interface crack problems.

Once singularity exponents are evaluated, either analytically or using a numerical method, the solution of the corner problem is obtained in an explicit form. This corner solution can be subsequently applied in a numerical solution of an actual problem at a non-smooth domain as a local asymptotic solution in order to improve accuracy and order of convergence of the solution.

Note that the method developed here for piecewise homogeneous anisotropic potential problems can be used directly for computation of singularity exponents in an anisotropic Helmholtz equation and in non-homogeneous potential problems with smooth variation of material properties, which appear in applications of the so-called Functionally Graded Materials (FGM's). As follows from the anisotropic uncoupled thermo-elasticity theory (Ting, 1996), the singularity exponents associated to a corner heat transfer problem, which can be obtained by the method developed in this work, represent, when shifted by +1, a subset of all singularity exponents of thermal stresses. (See Yosibash (1998) for a numerical analysis of thermal stress singularities using knowledge of heat flux singularities at a corner.)

An extension of the approach introduced here to cases where the homogeneous convection and radiation boundary conditions are prescribed on one or both corner faces and to cases where transmission conditions at some or all interfaces are not perfect (i.e. some resistance to flux takes place, and therefore the normal flux through the interface is proportional to the difference of potential at the adjacent wedges) should also be carried out developing, for example, Sinclair's (1980) approach. Also an analysis of the existence of power-logarithmic singularities in flux in the case of double eigenvalues (typically situated on the transition loci separating regions of real and complex eigenvalues associated to closed anisotropic multi-material corners which do not fulfill condition (F.4)) ought to be developed using approaches presented by Dempsey and Sinclair (1979), Dempsey (1995), Leguillon and Sanchez-Palencia (1987) and Ting (1996).

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Appendix A. Transformation of the unit normal vector to a line

Consider the unit tangent and normal vectors to a line $\Gamma \subset R^2$ at a point $\mathbf{x}$ which are related by $\mathbf{n} = \mathbf{Es}$, where $\mathbf{E}$ is the unit antisymmetric matrix, $E_{12} = -E_{21} = 1$ and $E_{11} = E_{22} = 0$. Note that $\mathbf{E}^{\mathrm{T}}\mathbf{E} = \mathbf{E}\mathbf{E}^{\mathrm{T}} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

It is obvious that the unit tangential vector to the transformed line $\widetilde{\Gamma} = \mathbf{L}\Gamma$ at $\tilde{\mathbf{x}}$ is given as

$$\tilde{\mathbf{s}} = \frac{\mathbf{Ls}}{\|\mathbf{Ls}\|} = \frac{\mathbf{Ls}}{\sqrt{\mathbf{s}^{\mathrm{T}}\mathbf{K}^{-1}\mathbf{s}}}, \tag{A.1}$$

where $(4)_1$ has been used. Starting from relation $\tilde{\mathbf{n}} = \mathbf{E}\tilde{\mathbf{s}}$ and using $\mathbf{s} = \mathbf{E}^{\mathrm{T}}\mathbf{n}$ leads to

$$\tilde{\mathbf{n}} = \frac{\mathbf{E}\mathbf{L}\mathbf{E}^{\mathrm{T}}\mathbf{n}}{\sqrt{\mathbf{n}^{\mathrm{T}}\mathbf{E}\mathbf{K}^{-1}\mathbf{E}^{\mathrm{T}}\mathbf{n}}} = \frac{(\mathbf{L}^{-1})^{\mathrm{T}}\mathbf{n}}{\sqrt{\mathbf{n}^{\mathrm{T}}\mathbf{Kn}}}, \tag{A.2}$$

where the well-known relation $\mathbf{E}\mathbf{A}\mathbf{E}^{\mathrm{T}} = |\mathbf{A}|(\mathbf{A}^{-1})^{\mathrm{T}}$, valid for any regular matrix $\mathbf{A} \in R^{2\times 2}$, has been used.

Appendix B. A particular case of transformation $\tilde{\mathbf{x}} = \mathbf{Lx}$

A symmetric positive definite matrix $\mathbf{K} \in R^{2\times 2}$ can be decomposed in the following way, called the symmetric real Schur decomposition (Golub and Van Loan, 1991):

$$\mathbf{K} = \mathbf{V}\mathbf{D}\mathbf{V}^{\mathrm{T}} = (\mathbf{V}\mathbf{D}^{1/2})(\mathbf{V}\mathbf{D}^{1/2})^{\mathrm{T}} = \mathbf{L}^{-1}(\mathbf{L}^{-1})^{\mathrm{T}}, \tag{B.1}$$

where $\mathbf{D} = \mathrm{diag}[k_1, k_2]$ is a diagonal matrix of eigenvalues of $\mathbf{K}$ $(k_1 \geqslant k_2 > 0)$

$$k_1, k_2 = \frac{1}{2}\left(K_{11} + K_{22} \pm \sqrt{(K_{11} - K_{22})^2 + 4K_{12}^2}\right), \tag{B.2}$$

and $\mathbf{V}$ is an orthogonal matrix with columns defined by the eigenvectors of $\mathbf{K}$, associated to the principal axes of material orthotropy, (x_1^*, x_2^*)

$$\mathbf{V} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \tag{B.3}$$

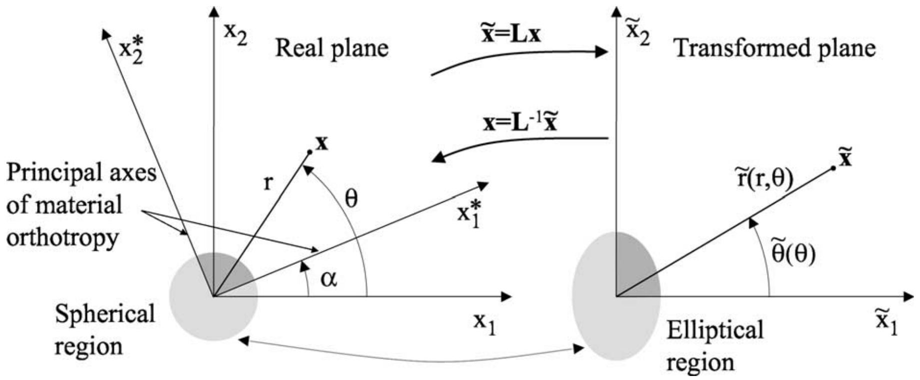

Fig. 5. Transformation of coordinates.

α being the angle of the first eigenvector with respect to axis x_1, see Fig. 5, given by

$$\alpha = \arctan\left(\frac{2K_{12}}{K_{11} - K_{22} + \sqrt{(K_{11} - K_{22})^2 + 4K_{12}^2}}\right), \tag{B.4}$$

where in the case of an indeterminate expression for $K_{12} = 0$, the above angle α can be taken $\alpha = \pi/2$ if $K_{11} < K_{22}$ and $\alpha = 0$ if $K_{11} = K_{22}$. Hence,

$$\mathbf{L} = \mathbf{D}^{-(1/2)}\mathbf{V}^{\mathrm{T}} = \begin{pmatrix} \dfrac{\cos\alpha}{\sqrt{k_1}} & \dfrac{\sin\alpha}{\sqrt{k_1}} \\ -\dfrac{\sin\alpha}{\sqrt{k_2}} & \dfrac{\cos\alpha}{\sqrt{k_2}} \end{pmatrix}. \tag{B.5}$$

Considering the polar coordinate system (r, θ) at the origin of the coordinates, the transformed radius vector and its length can be written using (B.5) as follows:

$$\tilde{\mathbf{x}} = r\left(\frac{\cos(\theta - \alpha)}{\sqrt{k_1}}, \frac{\sin(\theta - \alpha)}{\sqrt{k_2}}\right), \tag{B.6}$$

$$\tilde{r} = r\rho(\theta) = r\sqrt{k_1^{-1}\cos^2(\theta - \alpha) + k_2^{-1}\sin^2(\theta - \alpha)}. \tag{B.7}$$

Angle $\tilde{\theta}$ between the radius vector $\tilde{\mathbf{x}} = \tilde{r}\mathbf{e}(\tilde{\theta})$ and the coordinate axis $\tilde{x}_1$ can be evaluated by the following compact analytical expression:

$$\tilde{\theta}(\theta) = \begin{cases} \theta - \alpha & \text{for } \theta - \alpha = (2k+1)\dfrac{\pi}{2} \\ \left[\dfrac{\theta - \alpha}{\pi} + \dfrac{1}{2}\right]\pi + \arctan\left(\sqrt{\dfrac{k_1}{k_2}}\tan(\theta - \alpha)\right) & \text{for } \theta - \alpha \neq (2k+1)\dfrac{\pi}{2} \end{cases}, \tag{B.8}$$

where $[\cdot]$ means the integer part of a real number and $k \in Z$ is an integer number. Note that $\tilde{\theta}(\alpha) = 0$.

Differentiation of (B.8) results in

$$\frac{d\tilde{\theta}}{d\theta} = \frac{1}{\sqrt{k_1 k_2}\rho^2(\theta)}. \tag{B.9}$$

Appendix C. Relation between different transformations

Consider two different transformations defined by $\mathbf{L}_1$ and $\mathbf{L}_2$ with $|\mathbf{L}_1| > 0$ and $|\mathbf{L}_2| > 0$,

$$\mathbf{K}^{-1} = \mathbf{L}_1^{\mathrm{T}}\mathbf{L}_1 = \mathbf{L}_2^{\mathrm{T}}\mathbf{L}_2. \tag{C.1}$$

Taking into account that $\mathbf{L}_1$ is a regular matrix gives, via multiplication of this equality from the left and right, respectively, by $(\mathbf{L}_1^{\mathrm{T}})^{-1}$ and $\mathbf{L}_1^{-1}$,

$$(\mathbf{L}_2\mathbf{L}_1^{-1})^{\mathrm{T}} \cdot (\mathbf{L}_2\mathbf{L}_1^{-1}) = \mathbf{I}, \tag{C.2}$$

where $\mathbf{I}$ is the identity matrix. Defining $\mathbf{Q} = \mathbf{L}_2\mathbf{L}_1^{-1}$ Eq. (C.2) writes as

$$\mathbf{Q}^{\mathrm{T}}\mathbf{Q} = \mathbf{I}, \tag{C.3}$$

which means that the matrix $\mathbf{Q}$ is an orthogonal matrix (Golub and Van Loan, 1991) relating both transformations

$$\mathbf{L}_2 = \mathbf{Q}(\beta)\mathbf{L}_1, \quad \mathbf{Q}(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}. \tag{C.4}$$

Appendix D. General relation between θ and $\tilde{\theta}$

Let us evaluate the Jacobian of the transformation between the polar coordinate systems (r, θ) and $(\tilde{r}, \tilde{\theta})$. Direct evaluation gives

$$\frac{\partial(\tilde{r}, \tilde{\theta})}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial \tilde{r}}{\partial r} & \dfrac{\partial \tilde{r}}{\partial \theta} \\ \dfrac{\partial \tilde{\theta}}{\partial r} & \dfrac{\partial \tilde{\theta}}{\partial \theta} \end{vmatrix} = \frac{\partial \tilde{r}}{\partial r}\frac{\partial \tilde{\theta}}{\partial \theta} = \rho(\theta)\frac{\partial \tilde{\theta}}{\partial \theta}, \tag{D.1}$$

where the fact that $\partial\tilde{\theta}/\partial r = 0$ and (10) have been used. On the other hand, evaluation of this Jacobian by an application of the chain rule gives

$$\frac{\partial(\tilde{r}, \tilde{\theta})}{\partial(r, \theta)} = \frac{\partial(\tilde{r}, \tilde{\theta})}{\partial(\tilde{x}_1, \tilde{x}_2)}\frac{\partial(\tilde{x}_1, \tilde{x}_2)}{\partial(x_1, x_2)}\frac{\partial(x_1, x_2)}{\partial(r, \theta)} = \frac{1}{\tilde{r}}|\mathbf{L}|r = \frac{1}{\sqrt{|\mathbf{K}|}\rho(\theta)}, \tag{D.2}$$

where relations $(5)_2$ and (10) have been used. Equating the last expressions in (D.1) and (D.2) yields

$$\frac{d\tilde{\theta}}{d\theta} = \frac{1}{\sqrt{|\mathbf{K}|}\rho^2(\theta)}. \tag{D.3}$$

Appendix E. Proof that singularity exponents are real in open and closed isotropic multi-material corners

Let a piecewise constant function $\varkappa(\theta)$ be defined in the open interval (θ_0, θ_m) by positive numbers $\varkappa_n$ as follows: $\varkappa(\theta) = \varkappa_n > 0$ for $\theta \in (\theta_{n-1}, \theta_n)$, $n = 1, \ldots, m$, where $\theta_0 < \theta_1 < \cdots \theta_n \cdots < \theta_m$. Consider a space V of continuous functions in the closed interval $[\theta_0, \theta_m]$ defined as follows:

$$V = \left\{ f \in C^0[\theta_0, \theta_m] | f|_{[\theta_{n-1}, \theta_n]} \in C^2(\theta_{n-1}, \theta_n) \cap C^1[\theta_{n-1}, \theta_n]; \varkappa_n f'(\theta_n^-) = \varkappa_{n+1} f'(\theta_n^+); 1 \leqslant n \leqslant m-1 \right\}, \tag{E.1}$$

where C^k denotes k-times continuously differentiable functions in a set, and $f'(\theta_n^-)$ and $f'(\theta_n^+)$, respectively, denote the unilateral limits from the left and right of the first order derivative of f at θ_n.

Consider now a subspace V_0 of V where eigenfunctions associated to a multi-material corner problem (with homogeneous boundary conditions in the case of an open corner) will be looked for. Thus

$$V_0 = \left\{ f \in V | \text{b.c} \right\}, \tag{E.2}$$

where 'b.c.' denotes either $f'(\theta_0^+) = f'(\theta_m^-) = 0$, or $f(\theta_0^+) = f(\theta_m^-) = 0$, or $f'(\theta_0^+) = f(\theta_m^-) = 0$, or $f(\theta_0^+) = f'(\theta_m^-) = 0$ in the case of an *open corner*, or $f(\theta_0^+) = f(\theta_m^-)$ and $\varkappa_1 f'(\theta_0^+) = \varkappa_m f'(\theta_m^-)$ in the case of a *closed corner*.

Define a weighted scalar product for square integrable functions on (θ_0, θ_m) as follows:

$$\langle f, g \rangle_\varkappa \stackrel{\text{def}}{=} \int_{\theta_0}^{\theta_m} \varkappa(\theta) f(\theta) g(\theta)\, \mathrm{d}\theta. \tag{E.3}$$

It is obvious that $\langle f, f \rangle_\varkappa \geqslant 0$, and that $\langle f, f \rangle_\varkappa = 0$ if and only if f is zero almost everywhere on (θ_0, θ_m).

The following results will be required. If $f, g \in V_0$ then

$$\langle f'', g \rangle_\varkappa = -\langle f', g' \rangle_\varkappa. \tag{E.4}$$

Evaluation of the left-hand side in (E.4) by partitioning the integral to integrals on subintervals and integrating by parts yields

$$\begin{aligned}\langle f'', g \rangle_\varkappa &= \sum_{n=1}^{m} \int_{\theta_{n-1}}^{\theta_n} \varkappa_n f''(\theta) g(\theta)\, \mathrm{d}\theta = \sum_{n=1}^{m} \left((\varkappa_n f'(\theta) g(\theta))|_{\theta_{n-1}}^{\theta_n} - \int_{\theta_{n-1}}^{\theta_n} \varkappa_n f'(\theta) g'(\theta)\, \mathrm{d}\theta \right) \\ &= \sum_{n=1}^{m-1} (\varkappa_n f'(\theta_n^-) - \varkappa_{n+1} f'(\theta_n^+)) g(\theta_n) - \varkappa_1 f'(\theta_0^+) g(\theta_0) + \varkappa_m f'(\theta_m^-) g(\theta_m) - \langle f', g' \rangle_\varkappa. \end{aligned} \tag{E.5}$$

Taking into account the continuity conditions given in (E.1) and definitions of 'b.c' in (E.2), respectively, it is seen that the value of the sum in the second row of (E.5) and value of expression $-\varkappa_1 f'(\theta_0^+) g(\theta_0) + \varkappa_m f'(\theta_m^-) g(\theta_m)$ are zero, which proves (E.4).

Using (E.4) it can be shown that if $f \in V_0$ and

$$f''(\theta) + \lambda^2 f(\theta) = 0 \quad \text{for } \theta \in (\theta_{n-1}, \theta_n), \ n = 1, \ldots, m, \tag{E.6}$$

then λ is a real number. A straightforward calculation yields:

$$\lambda^2 \langle f, f \rangle_\varkappa = \langle -f'', f \rangle_\varkappa = \langle f', f' \rangle_\varkappa \geqslant 0, \tag{E.7}$$

where (E.6) has been used in obtaining the first equality and (E.4) in the second one. Thus, $\lambda^2 \geqslant 0$, consequently λ is a real number.

The main result of this Section can be stated as follows. Let $u(r, \theta) = r^\lambda f(\theta)$ be a solution of Laplace equation in each wedge $\Omega_n = \{\mathbf{x} \in R^2 | 0 < r < R(\theta), \theta_{n-1} < \theta < \theta_n\}$, $n = 1, \ldots, m$,

$$\varkappa_n \Delta u(r, \theta) = 0, \tag{E.8}$$

and the following interface conditions hold for $n = 1, \ldots, m-1$

$$u(r, \theta_n^-) = u(r, \theta_n^+), \tag{E.9}$$

$$q_n(r, \theta_n^-) = \varkappa_n \frac{1}{r} \frac{\partial u}{\partial \theta}(r, \theta_n^-) = \varkappa_{n+1} \frac{1}{r} \frac{\partial u}{\partial \theta}(r, \theta_n^+) = q_{n+1}(r, \theta_n^+) \tag{E.10}$$

and, additionally, either some of the following homogeneous boundary conditions $q_1(r, \theta_0^+) = 0 = q_m(r, \theta_m^-)$, or $u(r, \theta_0^+) = 0 = u(r, \theta_m^-)$, or $q_1(r, \theta_0^+) = 0 = u(r, \theta_m^-)$, or $u(r, \theta_0^+) = 0 = q_m(r, \theta_m^-)$, or the two closure conditions $u(r, \theta_0^+) = u(r, \theta_m^-)$ and $q_1(r, \theta_0^+) = q_m(r, \theta_m^-)$, hold. Then λ is a real number.

The proof starts by considering that $\varkappa_n \Delta u(r,\theta) = \varkappa_n r^{\lambda-2}(f''(\theta) + \lambda^2 f(\theta))$, $q_n(r,\theta) = r^{\lambda-1}\varkappa_n f'(\theta)$, and that a general solution of $f''(\theta) + \lambda^2 f(\theta) = 0$ is of the form $f(\theta) = a_c \cos\theta + a_s \sin\theta$ which is a bounded and smooth function for all θ. Then, the interface and boundary conditions for $u(r,\theta)$, respectively, are equivalent to continuity conditions on $f(\theta)$ in (E.1) and 'b.c.' in (E.2), and consequently $f \in V_0$ and (E.6) holds. Therefore, λ is a real number.

Note that system (E.6), (E.9) and (E.10) together with the boundary or closure conditions correspond to a Sturm–Liouville eigenvalue problem with piecewise constant coefficients discussed by Kellog (1971) and Birkhoff (1972).

Appendix F. Proof that singularity exponents are real in open anisotropic multi-material corners

Consider a multi-material [2] anisotropic corner Ω as studied in Section 4. The objective of this section is to show that any open anisotropic multi-material corner problem can be considered to be equivalent to an open isotropic multi-material corner problem. Then, the main result of the Appendix E can be applied to the anisotropic case considered here. As will be seen, the same statement holds for a particular case of closed anisotropic corner problems as well.

In order to obtain a mapping between an anisotropic and a corresponding isotropic multi-material corner which is continuous across interfaces between adjacent wedges, linear mapping used in (5) has to be modified when applied to the nth wedge by a scaling factor and by a rotation matrix defined, for example, by the following recursive formulae for $n = 1, \ldots, m-1$:

$$l_1 = 1 \quad l_{n+1} = l_n \frac{\rho_n(\theta_n)}{\rho_{n+1}(\theta_n)} \tag{F.1}$$

$$\mathbf{Q}_1 = \mathbf{I} \quad \mathbf{Q}_{n+1} = \mathbf{Q}\left(\tilde{\theta}_n(\theta_n) - \tilde{\theta}_{n+1}(\theta_n)\right)\mathbf{Q}_n, \tag{F.2}$$

where $\rho_n(\theta)$ is defined in (11), function $\tilde{\theta}_n(\theta)$ in (B.8), and $\mathbf{Q}(\cdot)$ in (C.4). Let $\mathscr{L}_n = l_n \mathbf{Q}_n \mathbf{L}_n$, $\mathbf{L}_n$ being given in (B.5). Then $\mathscr{L}_n^{\mathrm{T}} \mathscr{L}_n = l_n^2 \mathbf{K}_n^{-1}$ and $\mathscr{L}_n^{-1}(\mathscr{L}_n^{-1})^{\mathrm{T}} = l_n^{-2}\mathbf{K}_n$. Consider a mapping defined for $\mathbf{x} \in \overline{\Omega}_n$ (where $\overline{\Omega}_n$ denotes the closure of Ω_n) by $\mathscr{L}_n \mathbf{x} = \hat{\mathbf{x}}$ $(n = 1, \ldots, m)$. Denote the transformed nth wedge as $\mathscr{L}_n \Omega_n = \hat{\Omega}_n = \{\hat{\mathbf{x}} \in R^2 | 0 < \hat{r} < \hat{R}(\hat{\theta}), \hat{\theta}_{n-1} < \hat{\theta} < \hat{\theta}_n\}$.

It is not difficult to check that the set of linear mappings defined in this way represents a continuous mapping across the wedge interfaces. Actually, the transformed radius of a point $\mathbf{x}(r, \theta_n)$ at the interface between Ω_n and Ω_{n+1} is given from the side of $\hat{\Omega}_n$ by $\hat{r}_n(r,\theta_n) = r l_n \rho_n(\theta_n)$ and from the side of Ω_{n+1} by $\hat{r}_{n+1}(r,\theta_n) = r l_{n+1}\rho_{n+1}(\theta_n)$. Then, in view of (F.1)

$$\frac{\hat{r}_{n+1}(r,\theta_n)}{\hat{r}_n(r,\theta_n)} = \frac{l_{n+1}\rho_{n+1}(\theta_n)}{l_n \rho_n(\theta_n)} = 1 \tag{F.3}$$

for $n = 1, \ldots, m-1$. In a similar way, using (F.2), it can be shown that $\hat{\theta}_n(\theta_n) = \hat{\theta}_{n+1}(\theta_n)$ for $n = 1, \ldots, m-1$.

It has to be pointed out that condition $\hat{\theta}_m - \hat{\theta}_0 \leqslant 2\pi$ no longer has to be fulfilled.

In the case of a closed anisotropic multi-material corner, it is important to observe that $\hat{r}_m(r,\theta_m) = \hat{r}_1(r,\theta_0)$ if and only if $l_m \rho_m(\theta_m) = l_1 \rho_1(\theta_0)$, which according to (F.1) is equivalent to

$$\prod_{n=1}^{m} \frac{\rho_n(\theta_n)}{\rho_n(\theta_{n-1})} = 1. \tag{F.4}$$

[2] Magnitudes associated to the nth wedge are usually denoted either by a subscript 'n' or sometimes for the sake of clarity by '(n)'.

Note that this always holds for isotropic multi-material corners. Nevertheless, condition (F.4) does not hold, in general, for anisotropic multi-material corners. It is also interesting to observe the relation between condition (F.4) and (44) and (49).

Let u and $\hat{u}$ be, respectively, defined in Ω_n and $\hat{\Omega}_n$ and related by $u(\mathbf{x}) = \hat{u}(\mathscr{L}_n\mathbf{x})$ for $\mathbf{x} \in \Omega_n$. Then, analogously to (6)

$$K_{(n)ij}\partial_{x_i}\partial_{x_j}u(x) = K_{(n)ij}\mathscr{L}_{(n)ki}\mathscr{L}_{(n)lj}\partial_{\hat{x}_k}\partial_{\hat{x}_l}\hat{u}(\hat{\mathbf{x}}) = l_n^2\Delta\hat{u}(\hat{\mathbf{x}}). \tag{F.5}$$

Thus, fulfilling governing anisotropic Eq. (2) in Ω_n by u is equivalent to fulfilling Laplace equation by $\hat{u}$ in $\widehat{\Omega}_n$.

Define transformed flux vector associated to an isotropic material in $\widehat{\Omega}_n$ as

$$\hat{h}_i(\hat{\mathbf{x}}) \stackrel{\text{def}}{=} \varkappa_{\text{n}}\partial_{\hat{\text{x}}_\text{i}}\hat{\text{u}}(\hat{\mathbf{x}}), \tag{F.6}$$

where $\varkappa_n = \sqrt{|\mathbf{K}_n|}$. Then, as in (7) the following expression of the transformed flux vector is obtained:

$$\hat{h}_i(\hat{\mathbf{x}}) = \varkappa_n\partial_{\hat{x}_i}u(\mathscr{L}_n^{-1}\hat{\mathbf{x}}) = \varkappa_n u_{,j}(\mathbf{x})\mathscr{L}_{(n)ji}^{-1} = \varkappa_n l_n^{-2}\mathscr{L}_{(n)ij}h_j(\mathbf{x}), \tag{F.7}$$

where $h_i(\mathbf{x}) = K_{(n)ij}u_{,j}(\mathbf{x})$ for $\mathbf{x} \in \Omega_n$.

An analogous procedure to that used in Appendix A yields the following transformation rule for a unit normal vector to a curve in Ω_n

$$\hat{\mathbf{n}} = l_n\frac{(\mathscr{L}_n^{-1})^{\text{T}}\mathbf{n}}{n_{K_n}}, \quad \text{where } n_{K_n} = \sqrt{\mathbf{n}^{\text{T}}\mathbf{K}_n\mathbf{n}}. \tag{F.8}$$

Using expression (F.8) the following expression of the transformed normal flux in $\hat{\Omega}_n$ is deduced:

$$\hat{q}_n(\hat{\mathbf{x}}) = \hat{n}_i(\hat{\mathbf{x}})\hat{h}_i(\hat{\mathbf{x}}) = \varkappa_n\frac{q(\mathbf{x})}{l_n n_{K_n}(\mathbf{x})}. \tag{F.9}$$

Hence, when the transformed normal flux across the wedge interface at $\widehat{\theta}_n$ is evaluated, the following expressions are obtained:

$$\hat{q}_n(\hat{r}, \widehat{\theta}_n^-) = \varkappa_n\frac{q_n(r, \theta_n^-)}{l_n n_{K_n}(\theta_n)}, \quad \hat{q}_{n+1}(\hat{r}, \widehat{\theta}_n^+) = \varkappa_{n+1}\frac{q_{n+1}(r, \theta_n^+)}{l_{n+1}n_{K_{n+1}}(\theta_n)}. \tag{F.10}$$

According to (20) it holds that $n_{K_n}(\theta_n) = \varkappa_n\rho_n(\theta_n)$ and $n_{K_{n+1}}(\theta_n) = \varkappa_{n+1}\rho_{n+1}(\theta_n)$. Substituting these relations to (F.10) it is obtained, in view of (F.1), that the following equalities

$$\hat{q}_n(\hat{r}, \widehat{\theta}_n^-) = \hat{q}_{n+1}(\hat{r}, \widehat{\theta}_n^+) \quad \text{and} \quad q_n(r, \theta_n^-) = q_{n+1}(r, \theta_n^+) \tag{F.11}$$

are equivalent for $n = 1, \ldots, m-1$. Consequently any open anisotropic multi-material corner problem with its interface conditions of continuity in potential and fluxes is equivalent to an open isotropic multi-material corner problem. It has to be stressed that the same statement does not hold for closed anisotropic multi-material corner problems. It can be shown that a sufficient condition for this is (F.4), because if this condition holds then

$$\hat{q}_m(\hat{r}, \widehat{\theta}_m^-) = \hat{q}_1(\hat{r}, \widehat{\theta}_0^+) \quad \text{is equivalent to} \quad q_m(r, \theta_m^-) = q_1(r, \theta_0^+), \tag{F.12}$$

which can be shown in a similar way to (F.11).

Therefore, taking into account the main result of Appendix E, singularity exponents for open anisotropic multi-material corners, and also for closed corners providing (F.4) is fulfilled, are real numbers. An example of a closed bi-material anisotropic corner, which violates (F.4), with complex singularity exponents is given in Section 5.1.

References

Arad, M., Yosibash, Z., Ben-Dor, G., Yakhot, A., 1998. Computing flux intensity factors by a boundary method for elliptic equations with singularities. Communications in Numerical Methods in Engineering 14, 657–670.

Birkhoff, G., 1972. Angular singularities of elliptic problems. Journal of Approximation Theory 6, 215–230.

Clements, D., 1981. Boundary Value Problems Governed by Second Order Elliptic Systems. Pitman, London.

Costabel, M., Dauge, M., 1993. Construction of corner singularities for Agmon–Douglis–Nirenberg elliptic systems. Mathematische Nachrichten 162, 209–237.

Defourny, M., 1988. Singular point theory in Laplace field. In: Brebbia, C.A. (Ed.), Boundary Elements X, vol. 1. Springer Verlag, Berlin, pp. 165–180.

Dempsey, J.P., 1995. Power-logarithmic stress singularities at bi-material corners and interface cracks. Journal of Adhesion Science and Technology 9, 253–265.

Dempsey, J.P., Sinclair, G.B., 1979. On the stress singularities in the plane elasticity of composite wedge. Journal of Elasticity 9, 373–391.

Georgiu, G.C., Olson, L., Smyrlis, Y., 1996. A singular function boundary integral method for the Laplace equation. Communications in Numerical Methods in Engineering 12, 127–134.

Grisvard, P., 1992. Singularities in Boundary Value Problems. Masson/Springer Verlag, Paris/Berlin.

Golub, G.H., Van Loan, Ch.F., 1991. Matrix Computations. The John Hopkins University Press, Baltimore, MD.

Helsing, J., 2000. Corner singularities for elliptic problems: special basis functions versus 'brute force'. Communications in Numerical Methods in Engineering 16, 37–46.

Kellog, R.B., 1971. Singularities in interface problems. In: Hubbard, B. (Ed.), Numerical Solution of Partial Differential Equations—2: S.Y.N.S.P.A.D.E. 1970. Academic Press, New York, pp. 351–400.

Kondratev, V.A., 1967. Boundary-value problems for elliptic equations in domains with conical or angular points. Transactions of Moscow Mathematical Society 16, 227–313.

Lefeber, D., 1989. Solving Problems with Singularities using Boundary Elements. Computational Mechanics Publications, Southampton.

Leguillon, D., Sanchez-Palencia, E., 1987. Computation of Singular Solutions in Elliptic Problems and Elasticity. Masson, Paris.

Liggett, J.A., Liu, P.L.-F., 1983. The Boundary Integral Equation Method for Porous Media Flow. George Allen & Unwin, London.

Ma, C.C., Hour, B.L., 1989. Analysis of dissimilar anisotropic wedges subjected to antiplane shear deformation. International Journal of Solids and Structures 11, 1295–1309.

Ma, C.C., Hour, B.L., 1990. Antiplane problems in composite anisotropic materials with an inclined crack terminating at a bimaterial interface. International Journal of Solids and Structures 26, 1387–1400.

Mantič, V., París, F., 1995. On free terms and singular integrals in isotropic and anisotropic potential theory. In: Atluri, S.N., Yagawa, G., Cruse, T.A. (Eds.), Computational Mechanics '95. Springer Verlag, Berlin, pp. 2806–2811.

Mantič, V., París, F., Cañas, J., 1997. Stress singularities in 2D orthotropic corners. International Journal of Fracture 83, 67–90.

Mera, N.S., Elliott, L., Ingham, D.B., Lesnic, D., 2002. Singularities in anisotropic steady-state heat conduction using a boundary element method. International Journal for Numerical Methods in Engineering 53, 2413–2427.

Nazarov, S.A., Plamenevsky, B.A., 1994. Elliptic Problems in Domains with Piecewise Smooth Boundaries. Walter de Gruyter, Berlin.

Oh, H.-S., Babuška, I., 1995. The method of auxiliary mapping for the finite element solutions of elasticity problems containing singularities. Journal of Computational Physics 121, 193–212.

Pageau, S.S., Joseph, P.F., Biggers, B., 1995a. A finite element analysis of the singular stress fields in anisotropic materials loaded in antiplane shear. International Journal for Numerical Methods in Engineering 38, 81–97.

Pageau, S.S., Joseph, P.F., Biggers, B., 1995b. Singular antiplane stress fields for bonded and disbonded three-material junctions. Engineering Fracture Mechanics 52, 821–832.

Sinclair, G.B., 1980. On the singular eigenfunctions for plane harmonic problems in composite regions. Journal of Applied Mechanics 47, 87–92.

Szabó, B., Babuška, I., 1991. Finite Element Analysis. John Wiley, New York.

Ting, T.C.T., 1986. Explicit solution and invariance of the singularities at an interface crack in anisotropic composites. International Journal of Solids and Structures 22, 965–983.

Ting, T.C.T., 1996. Anisotropic Elasticity, Theory and Applications. Oxford University Press, Oxford.

Ting, T.C.T., 1997. Stress singularities at the tip of interfaces in polycrystals. In: Rossmanith, H.-P. (Ed.), Damage and Failure of Interfaces. Balkema, Rotterdam, pp. 75–82.

Wu, K.-C., 2001. Near-tip field and the associated path-independent integrals for anisotropic composite wedges. The Chinese Journal of Mechanics 17, 21–28.

Yosibash, Z., Szabó, B., 1995. Numerical analysis of singularities in two-dimensions Part 1: computation of eigenpairs. International Journal for Numerical Methods in Engineering 38, 2055–2082.

Yosibash, Z., 1998. Thermal generalized stress intensity factors in 2-D domains. Computer Methods in Applied Mechanics and Engineering 157, 365–385.